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On Graphs with Given Automorphism Group

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Given that a large graph admits a group of automorphisms isomorphic to the abstract group G, what is the probability $a(G)$ that G is its full automorphism group? This rational number can be computed from the structure of G. We determine the groups G with $a(G)=1$, show $a(G)=0$ or 1 for all abelian groups G, and observe that the values of $a(G)$ for metabelian groups G are dense in the unit interval.

1. STATEMENT OF RESULTS

An old theorem of Frucht [1] asserts that every finite group G is the automorphism group of a graph. Of course, it is trivial that there is a graph Γ with $G \leqslant \mathrm{Aut}(\Gamma)$; more difficult is to find such a graph and ensure that it has no extra automorphisms. In view of the well-known fact that almost all graphs have no non-trivial automorphisms, it is natural to pose the question: for which groups G is it true that almost all graphs Γ satisfying $G \leqslant \mathrm{Aut}(\Gamma)$ actually have $G = \mathrm{Aut}(\Gamma)$?

This question will be answered as a corollary of two more general theorems, which we now state. Given G, let $\mathscr{X}_n(G)$ be the class of isomorphism types of graphs Γ on n vertices with $G \leqslant \mathrm{Aut}(\Gamma)$. The statement "almost all graphs Γ with $G \leqslant \mathrm{Aut}(\Gamma)$ have property $\mathscr{P}$" is interpreted as meaning $\lim_{n\to\infty} |\mathscr{P} \cap \mathscr{X}_n(G)|/|\mathscr{X}_n(G)| = 1$.

THEOREM 1. *For almost all graphs Γ with $G \leqslant \mathrm{Aut}(\Gamma)$, the following assertions are true*:

(i) *the number of orbits of G on vertices of Γ is maximal (among all faithful G-spaces with the same number of points as vertices of Γ)*;

(ii) *subject to* (i), *the number of fixed points of G is minimal*;

(iii) *the orbits of G and* $\mathrm{Aut}(\Gamma)$ *on the vertices of Γ are the same.*

(Conclusion (i) may be interpreted to mean that Γ is as asymmetric as possible; (iii) restricts the extra automorphisms that Γ may have.)

Notation. Given a group G, let $p(X)$ and $q(X)$ denote the numbers of points and G-orbits in X, a faithful G-space with no fixed points. Let $X_1, \ldots, X_k$ be representatives for the isomorphism classes of such G-spaces X for which $p(X)-q(X)$ is minimal and (subject to this) $p(X)$ is maximal. (Note that there are only finitely many such isomorphism types.) For each X_i, consider the collection of graphs with vertex set X_i admitting G as a group of automorphisms. Identify two of the graphs in the collection (on the same or different vertex sets) if there is an isomorphism between them mapping G-orbits to G-orbits. Let $b(G)$ be the number of graphs in this collection $\mathscr{X}$, and $c(G)$ the number with the following property: any automorphism of the graph which fixes every G-orbit belongs to G. (Strictly speaking, the action of G on graphs in $\mathscr{X}$ is not well-defined, but the notion of "G-orbit" is meaningful, and $c(G)$ is the number of graphs for which the group of automorphisms fixing every "G-orbit" is isomorphic to G.) Let $a_n(G)$ be the proportion of those graphs Γ on n vertices with $G \leqslant \mathrm{Aut}(\Gamma)$ which satisfy $G = \mathrm{Aut}(\Gamma)$.

THEOREM 2. *For any finite group G,* $\lim_{n\to\infty} a_n(G) = a(G)$ *exists and is* $c(G)/b(G)$.

It is not known which rational numbers in $[0, 1]$ are the value of $a(G)$ for some group G.

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However, we can deduce some information from our theorem, including the answer to our earlier question "for which groups G do almost all graphs admitting G have no further automorphisms?"

COROLLARY 1. *$a(G)=1$ if and only if G is a direct product of symmetric groups.*

The groups of this corollary include the elementary abelian 2-groups. For abelian groups, we have a "zero-one law":

COROLLARY 2. *If G is abelian but not an elementary abelian 2-group, then $a(G)=0$.*

The next result is in sharp contrast.

COROLLARY 3. *The values of $a(G)$ for metabelian groups G are dense in $[0, 1]$.*

This is proved by computing $a(G)$ for some specific groups; the result has independent interest. Let ϕ and μ be the Euler and Möbius functions, and define

$$f(n)=\left(\sum_{d\mid n} 2^d\mu\left(\frac{n}{d}\right)\right)\Big/\left(\sum_{d\mid n} 2^d\phi\left(\frac{n}{d}\right)\right).$$

If n is prime, then $f(n)=s/(s+1)$, where $s=(2^{n-1}-1)/n$. The values of $f(n)$ for small n are given below.

n	2	3	4	5	6	7	8	9	10
$f(n)$	1/3	1/2	1/2	3/4	9/14	9/10	5/6	14/15	11/12

COROLLARY 4. *Let p be prime, m a divisor of $p-1$, and k a primitive mth root of unity mod p. Let G be the metacyclic group*

$$\langle x, y \mid x^p=y^m=1, y^{-1}xy=x^k\rangle$$

of order pm. Then

(i) *if m and p are odd, then $a(G)=0$;*
(ii) *if m is even and $m<p-1$, then $a(G)=f((p-1)/m)$;*
(iii) *if $m=p-1$ then $a(G)=1$ if $p=2$ or 3, 0 otherwise.*

2. PROOFS OF THE THEOREMS

We make use of the following facts:

(i) The number of isomorphism types of graphs on n vertices is $(2^{\frac{1}{2}n(n-1)}/n!)(1+O(1))$ (see [3, p. 196]).

(ii) The number of partitions of n is less than $2^{cn^{\frac{1}{2}}}$ for sufficiently large n, where $c=\pi\sqrt{\frac{2}{3}}\log e$ (see [2, p. 40]). (All logarithms are to the base 2.)

Let p be the number of points of a faithful permutation representation of G without fixed points, and q the number of orbits. We call a representation *special* if $p-q$ is minimal and (subject to this) p is maximal. The values of p and q for special representations are denoted by P and Q. Note that there are only finitely many special representations. Conclusions (i) and (ii) of Theorem 1 are equivalent to the assertion that the representation of G on its support is special.

The number of pairs (Γ, G^*), where Γ is a graph on n vertices, and G^* is a subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G and having a special representation on its support, is

$$\begin{aligned} F(n) &= (2^{\frac{1}{2}(n-P)(n-P-1)}/(n-P)!)2^{(n-P)Q}(c+o(1)) \\ &= (2^{\frac{1}{2}n(n-1-2(P-Q))}/(n-P)!)(c+o(1)); \end{aligned}$$

here and subsequently, c denotes a number depending on G but not on n, which may vary between occurrences.

We aim to show that, for almost all pairs (Γ, G^*) with $G^* \leq \mathrm{Aut}(\Gamma)$ and $G^* \cong G$, the representation of G^* on its support is special. It is convenient to show something stronger. Let λ be a partition of n, having q parts of size greater than 1, whose sum is p. Let $G(n)$ be the number of pairs (Γ, λ), where Γ is a graph on n vertices with a group G^* of automorphisms isomorphic to G, and λ is the partition of n induced by the orbits of a subgroup of $\mathrm{Aut}(\Gamma)$ containing G^*, with either $p-q > P-Q$, or $p-q = P-Q$, $p < P$. We show $G(n) = o(F(n))$.

To prove this, we split the partitions into four types.

(i) Those with $p-q = P-Q$ and $p < P$. Since $p \leq 2(P-Q)$, there is a bounded number of partitions to consider; the number of pairs (Γ, λ) is $O(2^{\frac{1}{2}n(n-1-2(P-Q))}/(n-p)!) = O(F(n)/n^{p-P})$.

(ii) Those with $p-q > P-Q$ and $p \leq m$ (where m is to be chosen). The number of partitions is bounded by $2^{cm^{\frac{1}{2}}}$. For each such partition, the number of graphs is at most

$$(2^{\frac{1}{2}n(n-1-2(p-q))+\frac{1}{2}m(m-1)})/(n-m)!.$$

So the number of pairs is $o(F(n))$ if m is sufficiently small, say $m \leq n^k$ where $k < \frac{1}{2}$.

(iii) Those with $m < p \leq \frac{1}{2}n$. Here there are at most $2^{cn^{\frac{1}{2}}}$ partitions, and for each partition there are at most

$$2^{\frac{1}{2}n(n-1)-(n-p)(p-q)} = O(F(n)\,.\,2^{n\log n-\frac{1}{4}mn})$$

graphs, since $p-q \geq \frac{1}{2}p > \frac{1}{2}m$. The number of pairs is thus $o(F(n))$ if m is sufficiently large, say $m \geq k \log n$, where $k > 4$.

(iv) Those with $p > \frac{1}{2}n$. Here a similar argument applies, to give at most $2^{\frac{1}{2}n(n-1)-\frac{1}{4}p(p-2)}$ graphs for each partition.

Choosing m to satisfy the requirements in (ii) and (iii), say $m = [n^{1/4}]$, we obtain the desired result.

Now let $\mathscr{G}$ be the following bipartite graph. Vertices of $\mathscr{G}$ are partitions of n for which either $p-q \geq P-Q$ or $p-q = P-Q$, $p \leq P$, together with isomorphism classes of graphs Γ on n vertices having $G^* \leq \mathrm{Aut}(\Gamma)$, $G^* \cong G$. Edges $\{\Gamma, \lambda\}$ are those for which λ is the partition induced by the orbits of a subgroup H of $\mathrm{Aut}(\Gamma)$ containing G^*. Special partitions are those with $p-q = P-Q$, $p = P$. We have seen that almost all edges of $\mathscr{G}$ contain a special partition. However, there is a fixed number of special partitions. So, for almost all graphs Γ with $G \leq \mathrm{Aut}(\Gamma)$, the orbits of $\mathrm{Aut}(\Gamma)$ form a special partition, whence G has a special representation on its support, and G and $\mathrm{Aut}(\Gamma)$ have the same orbits. This proves Theorem 1.

For Theorem 2, we may suppose that the support of G affords a special representation, and hence that the induced subgraph Δ on the support of G is a member of $\mathscr{X}$; we call Γ an *extension* of Δ. For any graph $\Delta \in \mathscr{X}$, the ratio of the number of graphs which extend Δ to $2^{\frac{1}{2}(n-P)(n-P-1)+(n-P)Q}/(n-P)!$ tends to 1 as $n \to \infty$; so the contributions of the graphs in $\mathscr{X}$ are all asymptotically equal. We may also suppose that G and $\mathrm{Aut}(\Gamma)$ have the same orbits. Since this forces $G = \mathrm{Aut}(\Gamma)$ for a proportion $c(G)/b(G)$ of the graphs in $\mathscr{X}$, the result follows.

3. Lemmata

The main difficulty in computing the number $a(G)$ is that of determining the special representations of G. The results of this section are directed towards this question.

LEMMA 1. *Let G be a group with a unique non-trivial minimal normal subgroup. Then the special representations of G are the faithful transitive representations of minimal degree.*

PROOF. In any faithful representation of G, the action on some orbit is faithful. Deleting any additional non-trivial orbits decreases the value of $p-q$.

LEMMA 2. *Let H and K be groups with the property that any special representation of $H\times K$ is a disjoint union of special representations of H and K. Suppose that one of the following holds:*

(i) $a(H)=a(K)=1$;

(ii) $a(H)=0$;

(iii) *there are no coincidences between orbit lengths in special representations of H and those for K.*

Then $a(H\times K)=a(H)a(K)$.

PROOF. The class $\mathscr{X}(H\times K)$ is obtained as follows: choose a graph in $\mathscr{X}(H)$ and one in $\mathscr{X}(K)$ and form the disjoint union; for each H-orbit and each K-orbit, include either all the edges between these orbits, or none. Now (i) and (ii) are clear. If (iii) holds, then each choice of a member of $\mathscr{X}(H)$ and a member of $\mathscr{X}(K)$ gives rise to $2^{q(H)q(K)}$ members of $\mathscr{X}(H\times K)$, since there can be no further identifications. The result follows.

REMARKS. 1. The lemma is not true without some assumption resembling (i)–(iii). For example, take $H=K=D_8$, the dihedral group of order 8. The only special representations of H are those of degree 4 (Lemma 1), and $\mathscr{X}(H)$ consists of the complete and null graphs, the cycle, and its complement; so $a(H)=a(K)=\frac{1}{2}$. A special representation of $H\times K$ has two orbits of length 4; each choice of two graphs from $\mathscr{X}(H)$ (equal or not) gives two members of $\mathscr{X}(H\times K)$. So $a(H\times K)=3/10$.

2. The hypothesis about special representations of $H\times K$ is not always satisfied. For example, let $G=C_2 \text{ wr } S_5$ (permutational wreath product) $=C_2\times K$, where K is a group of order $2^4 5!$. The special representations of both G and K are transitive of degree 10, and we have in fact $a(C_2)=1$, $a(K)=0$, $a(G)=\frac{1}{2}$. The next result gives hypotheses under which this pathology does not occur.

LEMMA 3. *Let H and K be groups such that*

(i) *any subnormal subgroup of H or K is normal;*

(ii) *any homomorphic image of $H\times K$ is a direct product of homomorphic images of H and K.*

Let X be an orbit in a special representation of $H\times K$. Then the group induced on X is a homomorphic image of either H or K.

PROOF. Suppose false and take a counter-example. By hypothesis (ii) and the fact that hypothesis (i) is preserved under homomorphisms, we may assume that the action on X is faithful.

Let h and k be the numbers of orbits of H and K respectively. Since K permutes the H-orbits in X transitively, and *vice versa*, the cardinality of the intersection of an H-orbit and a K-orbit is a constant m. Furthermore, the actions of H on its orbits are isomorphic, and similarly for K. Now the given representation of degree hkm can be replaced by an intransitive representation of degree $hm+km$, decreasing the value of $p-q$ unless one of h and k is 1.

Let us assume, then, that H is transitive (i.e. $h=1$). Then K is semi-regular ([5, p. 9]), so $|K|=m$. Let H_1 be the subgroup of H fixing each K-orbit, and H_2 the subgroup of H_1 fixing some K-orbit pointwise. Then $H_2 \trianglelefteq H_1 \trianglelefteq H$, whence $H_2 \trianglelefteq H$, whence $H_2=1$; that is, H_1 acts faithfully on each K-orbit. Again by [5, p. 9], H_1 is semi-regular, so $|H_1|\leqslant m$.

If $H_1 = 1$, then H acts faithfully on the set of K-orbits, so the given representation of degree km can be replaced by one of degree $k+m$. This requires $k=1$, whence $H=1$, a contradiction. So $H_1 \neq 1$.

Suppose $H_1 = m$. Then $Z(H_1) = H_1 \cap K = 1$. A group with trivial centre has a faithful permutation representation of degree at most half its order; so the given representation of degree km can be replaced by one of degree at most $\frac{1}{2}km + \frac{1}{2}m$, a contradiction. So $|H_1| < m$, and H_1 is not transitive on a K-orbit. But the stabiliser in H of a K-orbit is transitive on that orbit. This implies $k \geq 3$.

We may regard K as having its left regular representation on each orbit. Let $\bar{H}_1$ be the group acting as the right regular representation of K on each orbit. Then $|\bar{H}_1| = m$, $\bar{H}_1$ contains H_1, and $\bar{H} = H\bar{H}_1$ is a group. We are now in the situation of the previous paragraph, with one exception: we do not know that $Z(\bar{H}_1) = 1$, but only the weaker information that $Z(\bar{H}) \cap H_1 = 1$. However, an easy argument (which we omit) shows that a group possessing a non-trivial subgroup disjoint from its centre has a faithful permutation representation of degree at most three-quarters of its order. Thus the given representation of degree km can be replaced by one of degree at most $\frac{3}{4}km + \frac{3}{4}m$, contradicting $k \geq 3$.

REMARK. Hypothesis (ii) may not be necessary. Our earlier example shows that some version of hypothesis (i) is required, but it can be considerably weakened. For example, it is not difficult to show that $a(H \times K) = a(H)a(K)$ if H and K have coprime orders.

4. Proofs of the Corollaries

PROOF OF COROLLARY 1. The collection $\mathscr{X}$ of graphs always contains the null graph. So, if G has a special representation with orbit lengths $n_1, \ldots, n_k$, then the probability that $\mathrm{Aut}(\Gamma) = S_{n_1} \times \cdots \times S_{n_k}$ is positive. Thus, if $a(G) = 1$, then $G = S_{n_1} \times \cdots \times S_{n_k}$.

Conversely, suppose that G is such a direct product, and suppose first that none of the factors is S_4. Then Lemma 3 shows that the group induced on each orbit is a symmetric group. Since the representation is special, it has orbit lengths $n_1, \ldots, n_k$, and $a(G) = 1$. If some of the factors have degree 4, then hypothesis (i) of Lemma 3 fails, but an alternative argument (which we omit) applies.

PROOF OF COROLLARY 2. Let G be abelian. By Lemma 3, the group induced on each orbit is cyclic and regular. If $m_1, \ldots, m_k$ are the orbit lengths, then G is a subdirect product of $C_{m_1}, \ldots, C_{m_k}$, and hence a direct product of $C_{n_1}, \ldots, C_{n_k}$ with $n_i \leq m_i$ for each i. As the representation is special, $m_i = n_i$ for each i. If all m_i are equal to 2, then G has exponent 2. If $m_i > 2$, then any graph admitting G admits the dihedral group D_{2m_i} acting on the ith orbit, and $a(G) = 0$.

PROOF OF COROLLARY 4. Assume $p > 2$. The only special representation of G is the natural one of degree p (Lemma 1). The Sylow p-subgoup of G acts regularly, so any graph admitting G admits D_{2p}. Thus $a(G) = 0$ if m is odd, and we may suppose m is even. The last part is clear, so assume also $m < p-1$. Then G has $n = (p-1)/m$ orbits on edges of the complete graph, and any graph admitting G is a union of some of these orbits. Let S be the set of orbits.

Theorems of Galois and Burnside [5, p. 29] show that if $G \leq \mathrm{Aut}(\Gamma)$ and Γ is not complete or null, then G is a characteristic subgroup of $\mathrm{Aut}(\Gamma)$. Thus, any isomorphism between graphs admitting G is induced by the cyclic group C_n of permutations of S; and such a graph Γ has $G = \mathrm{Aut}(\Gamma)$ if and only if the corresponding subset is fixed by no non-trivial element of C_n.

The number $b(G)$ is thus the number of orbits of C_n on subsets of S, which is $1/n \sum_{d|n} 2^d \phi(n/d)$ by the Cauchy–Frobenius lemma (sometimes called Burnside's lemma [4; 5, p. 8]). Let $g(d)$ be the number of orbits of length d. This number is independent of n, and

$$2^n = \sum_{d|n} dg(d),$$

whence $c(G) = g(n) = 1/n \sum_{d|n} 2^d \mu(n/d)$. The result follows.

PROOF OF COROLLARY 3. Consider the function f of Corollary 4. The leading term in both numerator and denominator is 2^n; there are $o(n)$ further terms, each one $O(n2^{\frac{1}{2}n})$. So $f(n) \to 1$ as $n \to \infty$.

It follows from Dirichlet's theorem that there exist infinitely many metacyclic groups G with $a(G) = f(n)$ for any given n. (Choose primes $p \equiv 1 \pmod{2n}$.) Now, given $n_1, \ldots, n_k$, it is possible to choose such groups G_i with $a(G_i) = f(n_i)$ for $1 \leq i \leq k$, so that the chosen primes are all distinct and $p_i \nmid |G_j|$ for $i \neq j$. Application of Lemmas 2 and 3 now shows that $a(G_1 \times \cdots \times G_k) = f(n_1) \ldots f(n_k)$, and of course $G_1 \times \cdots \times G_k$ is metabelian. Given α, $\beta \in [0, 1]$ with $\alpha < \beta$, choose n such that $f(n) \in (1 + \alpha - \beta, 1)$, and k such that $f(n)^k \in (\alpha, \beta)$. The result follows.

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